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ON INDECOMPOSABLE ALGEBRAS OF EXPONENT 2*

BY

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ABSTRACT

For any $n \geq 3$ we give numerous examples of central division algebras of exponent 2 and index 2^n over fields, which do not decompose into a tensor product of two nontrivial central division algebras, and which are sums of $n+1$ quaternion algebras in the Brauer group of the field.

Also, for any $n \geq 3$ and any field k_0 we construct an extension F/k_0 and a multiquadratic extension L/F of degree 2^n such that for any proper subextensions L_1/F and L_2/F

$$W(L/F) \neq W(L_1/F)+W(L_2/F), \quad {}_2\operatorname{Br}(L/F) \neq {}_2\operatorname{Br}(L_1/F)+{}_2\operatorname{Br}(L_2/F).$$

1. Preliminaries

Let F be a field of characteristic not 2. A well-known result of Albert states that any central division algebra of index 4 and exponent 2 over F is a tensor product of two quaternion algebras [A]. In [T1] Tignol proved that any algebra of index 8 and exponent 2 is similar to a tensor product of four quaternion algebras. Moreover, by Merkurjev's theorem [M], any central simple algebra A of exponent 2 over a field is similar to a tensor product of quaternion algebras. In other words, any element $[A]$ of ${}_2\operatorname{Br} F$ is a sum of classes of quaternion algebras.

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If the index of A is 2^n, then, obviously, the number of summands in this sum is not less than n. If this number equals n, then by dimension count A itself is isomorphic to a tensor product of quaternion algebras. We call a central simple algebra over a field, decomposable, if it is a tensor product of two nontrivial central simple algebras over the same field. Otherwise, we say that the algebra is indecomposable. Obviously, if the algebra is indecomposable, it is necessarily a division algebra, i.e. a skewfield. The first example of indecomposable algebra of index 8 and exponent 2 was given in [ART]. In [K] Karpenko gave examples of indecomposable algebras of exponent 2^m and index 2^n for any $m \geq 1$, $n \geq 3$, $n \geq m$. An algebra A of such a type is obtained by means of the "generic" splitting of another algebra of index and exponent 2^n. However, if $\exp A = 2$ and $[A] = \sum_{i=1}^{p} [(a_i, b_i)]$ in these examples, it is unclear how small one can choose the number of summands p. In this paper we construct, for any field k_0 and $n \geq 3$, an indecomposable algebra of index 2^n over some field E/k_0, which is a sum of $n+1$ quaternion algebras in ${}_2\operatorname{Br} E$. One can choose the field E having no proper odd degree extension. Moreover, all the summands are split by the same prescribed multiquadratic field extension of degree 2^n over E. The description of the field E and the quaternion algebras is quite transparent.

We refer the reader to [Sch] as the main source concerning central simple algebras and quadratic forms over fields. The notation used in the sequel is more or less standard and coincides with the notation in [S]. All the fields in the paper are supposed to be of characteristic different from 2. If F is a field, then ${}_2\operatorname{Br} F$ stands for the 2-torsion of the Brauer group of F. Slightly abusing notation we will often identify a central simple algebra and its class in the Brauer group. $W(F)$ and $I(F)$ are respectively the Witt ring of F and the ideal of evendimensional quadratic forms in $W(F)$. By $K_2(F)$ we denote the Milnor's K_2 group of F. For $u, v \in F^*$ the symbol (u, v) denotes the quaternion algebra over F generated by the elements i and j with the relation

$$i^2 = u, \quad j^2 = v \quad ij = -ji.$$

If C is a projective conic over a field k, and p is a closed point of C, then $k(C)$ is the function field of C, and $k(p)$ is the residue field at p. If $f \in k(C)^*$, then (f) is the divisor of the function f. The abbreviations res, N, ind, deg denote respectively restriction, norm, index and degree. If L/F is a field extension and A is a central simple algebra over F, then by definition $A_L = \operatorname{res}_{L/F} A = A \otimes_F L$. We will write simply $A \otimes L$ instead of $A \otimes_F L$, or even merely A, if the restriction is clear from the context. The tensor product of central simple algebras over F is always considered over F. The sign $\sim$ means similarity of algebras, i.e., their equality in the Brauer group of the field.

2. Auxiliary and related results

The crucial point in the examples that we are going to give is some quaternion algebra, which was constructed in [S] and which is the basic tool in the proof of nonexcellency of multiquadratic field extensions in general. For the reader's convenience, we recall the construction of this algebra; moreover, that we need some additional properties.

Let k_0 be a field. Suppose n is a positive integer and $\overline{a}, \overline{b}_1, \dots, \overline{b}_n \in k_0^*/k_0^{*2}$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$.

Lemma 1: *There exist a tower of fields $k_0 \subset k_1 \subset \cdots \subset k_n$ and elements $x_i, y_i \in k_i^*$ for every $1 \le i \le n$ such that the following conditions hold:*

1) *The elements $\overline{a}, \overline{b}_1, \dots, \overline{b}_n$ remain linearly independent in k_n^*/k_n^{*2}.*
2) *Put $B_j = \sum_{i=1}^{j}(b_i, x_i + y_i\sqrt{a}) \in {}_2\operatorname{Br} k_j(\sqrt{a})$. Then for any $1 \le j \le n$* $\operatorname{ind} B_j = 2$.
3) *Put $A_j = \sum_{i=1}^{j}(b_i, x_i^2 - ay_i^2) \in {}_2\operatorname{Br} k_j$. Then for any $1 \le i \le j \le n$ and a finite field extension l/k_0 such that $(A_j)_{lk_j} = 0$, either $\sqrt{b_i} \in l$ or $\sqrt{ab_i} \in l$. In particular, $[l : k_0] \ge 2^j$.*

Proof. Let x_1, y_1 be indeterminates, $k_1 = k_0(x_1, y_1)$. Put

$$A_1 = (b_1, x_1^2 - ay_1^2) \in {}_2\operatorname{Br} k_1, \quad B_1 = (b_1, x_1 + y_1\sqrt{a}) \in {}_2\operatorname{Br} k_1(\sqrt{a}).$$

We will prove the lemma by induction on n. If $n = 1$, conditions 1) and 2) are obvious. To check condition 3) in this case, put $l_1 = l(\sqrt{a})(y_1)$ and consider the second residue map

$$\partial_{x_1 - \sqrt{a}y_1} : {}_2\operatorname{Br} l_1(x_1) \longrightarrow l_1^*/l_1^{*2}$$

with respect to the linear polynomial $x_1 - \sqrt{a}y_1$. Since $\partial_{x_1-\sqrt{a}y_1}((b_1, x_1^2 - ay_1^2)) = b_1$, we get that b_1 is a square in l_1, hence in $l(\sqrt{a})$, which is just what we need.

Now let $n \ge 2$. Assume that we have already constructed a tower of fields $k_0 \subset k_1 \subset \cdots \subset k_{n-1}$ and the algebras $A_1, \dots, A_{n-1}, B_1, \dots, B_{n-1}$ with the required conditions. Suppose $B_{n-1} = (u_1 + v_1\sqrt{a}, u_2 + v_2\sqrt{a})$, where $u_1, u_2, v_1, v_2 \in k_{n-1}^*$. Let t_1, t_2, t_3 be indeterminates, and $k_n = k_{n-1}(t_1, t_2, t_3)$. Let further $x_n, y_n \in k_n$

be such elements that

$$x_n + y_n\sqrt{a} = (u_1 + v_1\sqrt{a})t_1^2 + (u_2 + v_2\sqrt{a})t_2^2 - (u_1 + v_1\sqrt{a})(u_2 + v_2\sqrt{a})t_3^2,$$

i.e.,

$$x_n = u_1t_1^2 + u_2t_2^2 - (u_1u_2 + av_1v_2)t_3^2,$$
$$y_n = v_1t_1^2 + v_2t_2^2 - (u_1v_2 + u_2v_1)t_3^2.$$

Put

$$A_n = A_{n-1} + (b_n, x_n^2 - ay_n^2) \in {}_2\operatorname{Br} k_n,$$
$$B_n = B_{n-1} + (b_n, x_n + y_n\sqrt{a}) \in {}_2\operatorname{Br} k_n(\sqrt{a}).$$

It is easy to see that $x_n + y_n\sqrt{a}$ is a square in $(B_{n-1})_{k_{n-1}(\sqrt{a})}$, which implies that

$$(B_n)_{k_n(\sqrt{a})(\sqrt{x_n+y_n\sqrt{a}})} = 0,$$

hence $\operatorname{ind} B_n \le 2$. Obviously, $(B_{n-1})_{k_n(\sqrt{a})} \ne (b_n, x_n + y_n\sqrt{a})$, so $\operatorname{ind} B_n = 2$.

Now suppose that a finite extension l/k_0 is such that

$$0 = (A_n)_{lk_n} = (A_{n-1} + (b_n, x_n^2 - ay_n^2))_{lk_n}.$$

Then

$$0 = (A_n)_{lk_n(\sqrt{a})} = (A_{n-1})_{lk_n(\sqrt{a})} + (b_n, (x_n - y_n\sqrt{a})(x_n + y_n\sqrt{a}))_{lk_n(\sqrt{a})}.$$

Put $K = lk_{n-1}(\sqrt{a}, t_1, t_2)$. We can view the algebra

$$(A_{n-1})_{lk_n(\sqrt{a})} = (b_n, (x_n - y_n\sqrt{a})(x_n + y_n\sqrt{a}))_{lk_n(\sqrt{a})}$$

as an algebra over $K(t_3)$. It has no residues with respect to the second residue map

$$\partial = \bigoplus \partial_p : {}_2\operatorname{Br} K(t_3) \longrightarrow \coprod_p K(p)^*/K(p)^{*2},$$

where p runs over all monic irreducible polynomials from $K[t_3]$. On the other hand, the monic polynomials

$$P_1(t_3) = t_3^2 - \frac{(u_1 + v_1\sqrt{a})t_1^2 + (u_2 + v_2\sqrt{a})t_2^2}{(u_1 + v_1\sqrt{a})(u_2 + v_2\sqrt{a})} = \frac{-(x_n + y_n\sqrt{a})}{(u_1 + v_1\sqrt{a})(u_2 + v_2\sqrt{a})}$$

and

$$P_2(t_3) = t_3^2 - \frac{(u_1 - v_1\sqrt{a})t_1^2 + (u_2 - v_2\sqrt{a})t_2^2}{(u_1 - v_1\sqrt{a})(u_2 - v_2\sqrt{a})} = \frac{-(x_n - y_n\sqrt{a})}{(u_1 - v_1\sqrt{a})(u_2 - v_2\sqrt{a})}$$

are distinct. Hence

$$0 = \partial_{P_1}((A_{n-1})_{lk_n(\sqrt{a})}) = b_n,$$

and so,

$$\sqrt{b_n} \in lk_n\left(\sqrt{a}, t_1, t_2, \sqrt{\frac{(u_1+v_1\sqrt{a})t_1^2+(u_2+v_2\sqrt{a})t_2^2}{(u_1+v_1\sqrt{a})(u_2+v_2\sqrt{a})}}\right).$$

This obviously implies $\sqrt{b_n} \in lk_n(\sqrt{a})$. Therefore, either $\sqrt{b_n} \in lk_n$ or $\sqrt{ab_n} \in lk_n$. Since the extension k_n/k_0 is purely transcendental, we get that either $\sqrt{b_n} \in l$, or $\sqrt{ab_n} \in l$. In both cases it follows that $(b_n, x_n^2 - ay_n^2)_{lk_n} = 0$. Since

$$0 = (A_n)_{lk_n(\sqrt{a})} = (A_{n-1})_{lk_n(\sqrt{a})} + (b_n, x_n^2 - ay_n^2)_{lk_n(\sqrt{a})},$$

we get

$$(A_{n-1})_{lk_n(\sqrt{a})} = 0.$$

Since $k_n = k_{n-1}(t_1, t_2, t_3)$, it follows that

$$(A_{n-1})_{lk_{n-1}(\sqrt{a})} = 0,$$

and by the induction assumption we conclude that for any $1 \le i \le n-1$ either $\sqrt{b_i} \in l$ or $\sqrt{ab_i} \in l$. The induction step is done, so the lemma is proved. ∎

Since $\operatorname{ind} B_n = 2$, it follows that $\operatorname{ind} A_n = \operatorname{ind} N_{k_n(\sqrt{a})/k_n}(B_n) = 2$ or 4. If $\operatorname{ind} A_n = 2$, put $k = k_n$. If $\operatorname{ind} A_n = 4$ put $k = k_n(\phi)$, where ϕ is an Albert form corresponding to A_n. It is easy to see that condition 3) remains valid for the field k and the algebra $(A_n)_k$, i.e., if the extension l/k_0 is finite and $(A_n)_{lk} = 0$, then for any $1 \le i \le n$ either $\sqrt{b_i} \in l$ or $\sqrt{ab_i} \in l$.

Now put $A = (A_n)_k$, $B = (B_n)_{k(\sqrt{a})}$. Let C be the projective conic over k corresponding to A. Set $F = k(C)$. Then, obviously, $A_F = 0$. Consider the exact sequence [M]

$${}_2\operatorname{Br} F \xrightarrow{\text{res}} {}_2\operatorname{Br} F(\sqrt{a}) \xrightarrow{\text{N}} {}_2\operatorname{Br} F.$$

Since

$$0 = A_F = N_{F(\sqrt{a})/F}(B_{F(\sqrt{a})}),$$

we have $B_{F(\sqrt{a})} = D_{F(\sqrt{a})}$ for some $D \in {}_2\operatorname{Br} F$. Since $\operatorname{ind} B = 2$, we get $\operatorname{ind} D \le 4$. It is well-known that if $\operatorname{ind} D = 4$, then $D = D_1 + D_2$ for some $D_1, D_2 \in {}_2\operatorname{Br} F$, such that $\operatorname{ind} D_1 = \operatorname{ind} D_2 = 2$ and $D_{2F(\sqrt{a})} = 0$. Hence, we may change D for D_1 and so assume that $\operatorname{ind} D \le 2$.

From now on we will assume that $n \ge 2$. Put $L = F(\sqrt{a}, \sqrt{b_1}, \dots, \sqrt{b_n})$. Let $F \subset F_1 \subset F_2 \subset L$ be a tower of fields such that

$$\sqrt{a} \notin F_2, \quad [F_2 : F_1] = 4, \quad [L : F_1] = 8.$$

Obviously, $L = F_2(\sqrt{a}) = F_1(\sqrt{a})F_2$ and $F_1 = kl_1(C)$, $F_2 = kl_2(C)$ for some multiquadratic extensions l_1/k_0, l_2/k_0 of degree 2^{n-2} and 2^n, respectively.

Proposition 2: 1) *The elements* $\overline{a}, \overline{b_1}, \dots, \overline{b_n}$ *remain linearly independent in* F^*/F^{*2};

2) $\operatorname{ind} D = 2$;

3) $D_L = 0$;

4) *the algebra* D_{F_1} *does not decompose into a sum* $D_{F_1} = D_1 + D_2$, *where* $D_1 \in {}_2\operatorname{Br}(F_1(\sqrt{a})/F_1)$ *and* $D_2 \in {}_2\operatorname{Br}(F_2/F_1)$;

Proof. 1) Obvious.

2) Assume that $\operatorname{ind} D = 1$, i.e. $D = 0$. Then $B_{k(\sqrt{a})(C)} = 0$, hence $B_{k(\sqrt{a})}$ is either 0 or $A_{k(\sqrt{a})}$. In both cases $A = N_{k(\sqrt{a})/k} B = 0$, a contradiction.

3) Obvious, since $D_{F(\sqrt{a})} = \sum_{i=1}^{n} (b_i, x_i + y_i\sqrt{a})$.

4) Assume the contrary, i.e., that $D = D_1 + D_2$, where $D_1 \in {}_2\operatorname{Br}(F_1(\sqrt{a})/F_1)$ and $D_2 \in {}_2\operatorname{Br}(F_2/F_1)$. Then

$$B_{F_1(\sqrt{a})} = D_{F_1(\sqrt{a})} = D_{2F_1(\sqrt{a})}.$$

Denote the field kl_1 by $\widehat{k}$, so that $F_1 = \widehat{k}(C)$. Consider the points $z_1, \dots, z_m \in C_{\widehat{k}}$ at which the algebra D_2 has nonzero residues under the second residue map

$${}_2\operatorname{Br} F_1 \to \coprod_{z \in C_{\widehat{k}}} \widehat{k}(z)^* / \widehat{k}(z)^{*2}.$$

Since B is defined over $k(\sqrt{a})$, the algebra $D_{2F_1(\sqrt{a})} = B_{F_1(\sqrt{a})}$ has no residues at all. This implies that the residues of D_2 at the points $z_1, \dots, z_m$ are equal to a. Recall that $F_2 = kl_2(C)$, and let $c_1, \dots, c_n \in k_0^*$ be elements such that $l_2 = k_0(\sqrt{c_1}, \dots, \sqrt{c_n})$. Therefore, since $D_{2F_2} = 0$, we have for any j

$$a_{\widehat{k}(z_j)(\sqrt{c_1}, \dots, \sqrt{c_n})} \in \widehat{k}(z_j)(\sqrt{c_1}, \dots, \sqrt{c_n})^{*2}.$$

Denote by c_I the product $\prod_{i \in I} c_i$ (the similar notation will be used also in the sequel). Thus, given any j, we have $ac_I \in \widehat{k}(z_j)^{*2}$ for some $I \subset \{1, \dots, n\}$ depending on j, i.e.,

$$\widehat{k}(\sqrt{ac_I}) \subset \widehat{k}(z_j). \qquad ∎$$

Lemma 3: *For any* j *we have* $4 \mid \deg\, z_j$.

Proof. Assume this is not the case. Consider the tower $\widehat{k} \subset \widehat{k}(\sqrt{c}) \subset \widehat{k}(z_j)$, where $c = ac_I$ and I corresponds to the point z_j. Notice that $\sqrt{c} \notin \widehat{k} = kl_1$, for otherwise $\sqrt{c} \in kl_2$, and so $\sqrt{a} \in F_2$, a contradiction. Hence $[\widehat{k}(z_j) : \widehat{k}(\sqrt{c})]$ is odd. Since $A_{\widehat{k}(z_j)} = 0$ we conclude that $A_{\widehat{k}(\sqrt{c})} = 0$. Therefore, $(A_n)_{k_n l_1(\sqrt{c})} = 0$, a contradiction to Lemma 1, since $[l_1(\sqrt{c}) : k_0] = 2^{n-1}$. The lemma is proved. ∎

Now we use the fact that any divisor of degree zero on a projective conic is principal. Choose any $y \in C_{\widehat{k}}$ such that $\deg y = 2$. Let $s = \sum_{i=1}^{m} \deg z_j$. Consider the divisor $\mathfrak{a} = -(s/2)y + \sum_{i=1}^{m} z_i$. Obviously, $\deg \mathfrak{a} = 0$, hence there is $f \in F_1^*$ such that $\mathfrak{a} = (f)$. The algebra (a, f) has nonzero residues just at the points $z_1, \ldots, z_m$, since by Lemma 3 the number $s/2$ is even. Moreover, all these residues are equal to a. Therefore, the algebra $\widehat{D_2} = D_2 + (a, f)$ has no residues, i.e., $\widehat{D_2} \in {}_2\operatorname{Br}(C_{\widehat{k}})$. Merkurjev's theorem [M] claims that $K_2\widehat{k}(C)/2K_2\widehat{k}(C) \simeq {}_2\operatorname{Br}\widehat{k}(C)$. Hence by ([Su], Lemma 5) we get that either

$$\widehat{D_2} \in \operatorname{res}_{F_1/\widehat{k}}({}_2\operatorname{Br}\widehat{k})$$

or

$${}_2\operatorname{Br}(C_{\widehat{k}})/\operatorname{res}_{F_1/\widehat{k}}({}_2\operatorname{Br}\widehat{k}) = \mathbb{Z}/2\mathbb{Z},$$

and the element $\widehat{D_2}$ is nontrivial in this factor group.

In the first case let $\widehat{D_2} = \operatorname{res}_{F_1/\widehat{k}} \widetilde{D}$ for some $\widetilde{D} \in {}_2\operatorname{Br}\widehat{k}$. Then

$$\begin{aligned}(\widetilde{D} + B)_{\widehat{k}(\sqrt{a})(C)} = (\widehat{D_2} + B)_{\widehat{k}(\sqrt{a})(C)} &= (D_2 + B)_{\widehat{k}(\sqrt{a})(C)} \\ &= (D + B)_{\widehat{k}(\sqrt{a})(C)} = 0.\end{aligned}$$

Hence $(\widetilde{D} + B)_{\widehat{k}(\sqrt{a})}$ is either zero or $A_{\widehat{k}(\sqrt{a})}$. But then

$$0 = N_{\widehat{k}(\sqrt{a})/\widehat{k}}(\widetilde{D} + B) = N_{\widehat{k}(\sqrt{a})/\widehat{k}} B = A_{\widehat{k}},$$

hence $(A_n)_{l_1 k_n} = 0$, a contradiction to Lemma 1, since $[l_1 : k_0] = 2^{n-2}$. In the second case, when $\widehat{D_2} \notin \operatorname{res}_{F_1/\widehat{k}}({}_2\operatorname{Br}\widehat{k})$, let $A = (a_1, a_2)$, where $a_1, a_2 \in k^*$. Then F_1 is the quotient field of the ring $\widehat{k}[u_1, u_2]/(a_1u_1^2 + a_2u_2^2 - 1)$, where u_1, u_2 are indeterminates. It is easy to see that $N_{F_1/\widehat{k}(u_1)}(\widehat{D}_2) = A$. Consider the commutative diagram

$$\begin{array}{ccc} {}_2\operatorname{Br} F_1 & \xrightarrow{\operatorname{res}} & {}_2\operatorname{Br} F_1(\sqrt{a}) \\ \Big\downarrow N & & \Big\downarrow N \\ {}_2\operatorname{Br}\widehat{k}(u_1) & \xrightarrow{\operatorname{res}} & {}_2\operatorname{Br}\widehat{k}(\sqrt{a})(u_1) \end{array}$$

So we have

$$\begin{aligned}\operatorname{res}_{\widehat{k}(\sqrt{a})(u_1)/\widehat{k}(u_1)} A &= \operatorname{res}_{\widehat{k}(\sqrt{a})(u_1)/\widehat{k}(u_1)} \circ N_{F_1/\widehat{k}(u_1)}(\widehat{D_2}) \\ &= N_{F_1(\sqrt{a})/\widehat{k}(\sqrt{a})(u_1)} \circ \operatorname{res}_{F_1(\sqrt{a})/F_1}(\widehat{D_2}) = N_{F_1(\sqrt{a})/\widehat{k}(\sqrt{a})(u_1)}(B) = 0,\end{aligned}$$

hence $\operatorname{res}_{\widehat{k}(\sqrt{a})/k}(A) = 0$, a contradiction to Lemma 1, since $[l_1(\sqrt{a}) : k_0] = 2^{n-1}$. Thus Proposition 2 is proved. ∎

Recall that $L = F(\sqrt{a}, \sqrt{b_1}, \dots, \sqrt{b_n})$. Proposition 2 implies the following corollary, which is of some independent interest, and which can be considered as a generalization of Theorem 5.1 in [ELTW].

Corollary 4: *Let $F \subset L_1 \subset L$, $F \subset L_2 \subset L$ be proper field subextensions of L/F. Then*

1) *$D \notin {}_2\operatorname{Br}(L_1/F) + {}_2\operatorname{Br}(L_2/F)$. In particular,*

$$ {}_2\operatorname{Br}(L/F) \neq {}_2\operatorname{Br}(L_1/F) + {}_2\operatorname{Br}(L_2/F).$$

2) *$f \notin W(L_1/F) + W(L_2/F)$, where $f \in W(L/F)$ is the 2-fold Pfister form corresponding to D.*

Proof. 1) Obviously we may assume that $[L : L_1] = [L : L_2] = 2$ and $L = L_1L_2$. Then $[L : L_1 \cap L_2] = 4$. Choose a field F_1 such that

$$F \subset F_1 \subset L_1 \cap L_2, \quad \sqrt{a} \notin F_1^*, \quad [L_1 \cap L_2 : F_1] = 2.$$

Then

$$[L : F_1] = 8, \quad [L_1 : F_1] = [L_2 : F_1] = 4,$$
$$L_1 = F_1(\sqrt{d_1}, \sqrt{d_2}), \quad L_2 = F_1(\sqrt{d_1}, \sqrt{d_3})$$

for some d_1, d_2, d_3 belonging to the multiplicative group generated by the elements $a, b_1, \dots, b_n$. We will show that $D_{F_1} \notin {}_2\operatorname{Br}(L_1/F_1) + {}_2\operatorname{Br}(L_2/F_1)$. Assume the contrary. Then

$$D_{F_1} = (d_1, e_1) + (d_2, e_2) + (d_3, e_3)$$

for some $e_1, e_2, e_3 \in F_1^*$. Obviously, we may suppose that a is equal either to d_i, or to d_id_j $(i < j)$, or to $d_1d_2d_3$. We have

$$D_{F_1} = (d_1d_2, e_1) + (d_2, e_1e_2) + (d_3, e_3) = (d_1d_2d_3, e_1) + (d_2, e_1e_2) + (d_3, e_1e_3).$$

Put $F_2 = F_1(\sqrt{d_2}, \sqrt{d_3})$. We conclude that

$$D_{F_1} \in {}_2\operatorname{Br}(F_1(\sqrt{a})/F_1) + {}_2\operatorname{Br}(F_2/F_1),$$

which contradicts Proposition 2.

2) This part easily follows from part 1). Indeed, assume that $[f] = [f_1] + [f_2]$, where $f_1 \in W(L_1/F)$, $f_2 \in W(L_2/F)$. Obviously, $\mathrm{disc}(f_1) = \mathrm{disc}(f_2)$. Let $g_1, g_2 \in I^2(F)$ be forms, such that $\dim(f_i \perp -g_i)_{an} \leq 2$ for $i = 1, 2$. It is clear that $g_i \in W(L_i/F)$. If $D_i \in {}_2\mathrm{Br}\, F$ correspond to g_i under the map $I^2(F)/I^3(F) \simeq {}_2\mathrm{Br}\, F$, then $D = D_1 + (D_2 + (\mathrm{disc}\, f_2, u))$ for some $u \in F^*$, a contradiction to part 1) since $\langle\langle \mathrm{disc}(f_2)\rangle\rangle \in W(L_2/F)$. ∎

Lemma 5: *Let K be a field, $K((t))_{odd}$ any maximal odd degree extension of $K((t))$. Then the following holds:*

1) *there exists a field inclusion $K_{odd}((t)) \hookrightarrow K((t))_{odd}$ over $K((t))$ for some maximal odd degree extension K_{odd} of K;*
2) *${K((t))_{odd}}^*/{K((t))_{odd}}^{*2} \simeq {K_{odd}}^*/{K_{odd}}^{*2} \oplus \mathbb{Z}/2\mathbb{Z}$, where the element $t \in {K((t))_{odd}}^*/{K((t))_{odd}}^{*2}$ corresponds to the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ and ${K_{odd}}^*/{K_{odd}}^{*2}$ is naturally embedded into ${K((t))_{odd}}^*/{K((t))_{odd}}^{*2}$ under this isomorphism.*

Proof. 1) Any finite extension L of $K((t))$ is a complete valuation field of characteristic equal to the characteristic of its residue field. We have $L = \overline{L}((\pi))$, where $\overline{L}$ is a subfield of L isomorphic to the residue field and π is a uniformizer ([Se], Chapter 2, Theorem 2). It is easy to check that we can choose for K_{odd} the union of all finite extensions of K contained in $K((t))_{odd}$.

2) Assume $L/K_{odd}((t))$ is a finite odd degree extension. Then

$$L \simeq K_{odd}((\pi)), \quad L^*/{L^*}^2 \simeq {K_{odd}}^*/{K_{odd}}^{*2} \oplus \mathbb{Z}/2\mathbb{Z},$$

where $\pi \in L^*/{L^*}^2$ corresponds to the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ and ${K_{odd}}^*/{K_{odd}}^{*2}$ is naturally embedded into $L^*/{L^*}^2$. Denote by v the discrete valuation on L. Obviously, $v(t)$ is odd, hence $\pi t \in {K_{odd}}^*/{K_{odd}}^{*2} \hookrightarrow L^*/{L^*}^2$. Therefore, there exists another isomorphism

$$L^*/{L^*}^2 \simeq {K_{odd}}^*/{K_{odd}}^{*2} \oplus \mathbb{Z}/2\mathbb{Z},$$

where $t \in L^*/{L^*}^2$ corresponds to the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ and ${K_{odd}}^*/{K_{odd}}^{*2}$ is naturally embedded into $L^*/{L^*}^2$. Also it is clear that the last isomorphisms are compatible for various L, which completes the proof. ∎

Corollary 6: *Let K be a field, $E = K((t_0))\dots((t_n))$ the iterated Laurent series field. Let E_{odd} be any maximal odd degree extension of E. Then*

$$K_{odd} \hookrightarrow E_{odd}, \quad {E_{odd}}^*/{E_{odd}}^{*2} \simeq K^*_{odd}/{K^*_{odd}}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^{n+1}$$

for some maximal odd degree extension K_{odd} of K, and under this isomorphism the element t_i $(0 \le i \le n)$ corresponds to the nontrivial element of the $i+1$-st factor of $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ while $K^*_{odd}/{K^*_{odd}}^2$ is naturally embedded into $E^*_{odd}/{E^*_{odd}}^2$.

Proof. Taking into account Lemma 5, the proof is immediate by induction on n. ∎

Lemma 7: *Let K be a field, let $\overline{c_0},\dots,\overline{c_n} \in K^*/K^{*2}$ be linearly independent elements. Suppose the algebra $\alpha \in {}_2\operatorname{Br} K$ is such that $\alpha_{K(\sqrt{c_0},\sqrt{c_1},\dots,\sqrt{c_n})} \neq 0$. Let further $t_0,\dots,t_n,x$ be indeterminates. Then for any $0 \le i \le n+1$*

$$\operatorname{ind}[\alpha + (c_0,t_0) + \dots + (c_n,t_n)]_E \ge 2^{n+2},$$

where

$$E = K((t_0))\dots((t_{i-1}))((x))((t_i))\dots((t_n)).$$

Proof. The argument is by induction on n. Suppose first that $i \le n$. Put

$$E_1 = K((t_0))\dots((t_{i-1}))((x))((t_i))\dots((t_{n-1})).$$

By induction hypothesis $\operatorname{ind}[\alpha + (c_0,t_0) + \dots + (c_{n-1},t_{n-1})_{E_1(\sqrt{c_n})}] \ge 2^{n+1}$. Applying Tignol's theorem ([T2], Proposition 2.4) we finish the proof. If $i = n+1$, the argument is similar with a slight modification. ∎

3. Construction of indecomposable algebras

We turn now to construct the indecomposable algebras of an arbitrary 2-primary index and exponent 2. From now on we change the notation a bit.

Let F be an arbitrary field, $n \ge 2$, $a, b_1, \dots, b_n \in F^*$, $D \in {}_2\operatorname{Br} F$. We call the triple $(F, D, \{a, b_1, \dots, b_n\})$ **admissible** if the following conditions hold:

a) the elements $\overline{a}, \overline{b_1}, \dots, \overline{b_n} \in F^*/F^{*2}$ are linearly independent;

b) $\operatorname{ind} D = 2$;

c) $D_{F(\sqrt{a},\sqrt{b}_1,\dots,\sqrt{b}_n)} = 0$;

d) for any tower $F \subset F_1 \subset F_2 \subset L = F(\sqrt{a},\sqrt{b}_1,\dots,\sqrt{b}_n)$ such that

$$\sqrt{a} \notin F_2^*, \quad [L : F_1] = 8, \quad [F_2 : F_1] = 4$$

we have

$$D_{F_1} \notin {}_2\operatorname{Br}(F_1(\sqrt{a})/F_1) + {}_2\operatorname{Br}(F_2/F_1);$$

e) the field F has no proper extensions of odd degree.

Notice that Proposition 2 give us examples of triples satisfying conditions $a) - d)$ of the above definition.

PROPOSITION 8: 1) *If the triple $(F, D, \{a, b_1, \dots, b_n\})$ satisfies condition $a)-d)$ above, then the triple $(F_{odd}, D_{odd}, \{a, b_1, \dots, b_n\})$ is admissible.*

2) *If the triple $(F, D, \{a, b_1, \dots, b_n\})$ is admissible, $n \geq 3$, $c \in F^*$, and*

$$F(\sqrt{a}, \sqrt{b}_1, \dots, \sqrt{b}_n) = F(\sqrt{c}, \sqrt{a}, \sqrt{b_1}, \dots, \sqrt{b_{n-1}}),$$

then the triple $(F(\sqrt{c}), D_{F(\sqrt{c})}, \{a, b_1, \dots, b_{n-1}\})$ is admissible as well.

3) *If the triple $(F, D, \{a, b_1, \dots, b_n\})$ is admissible, and*

$$L = F(\sqrt{a}, \sqrt{b}_1, \dots, \sqrt{b}_n),$$

then Corollary 4 holds for the extension L/F and the algebra D.

Proof. 1) Conditions a), b), c) and e) are obvious, and applying the norm map we obtain condition d).

2) The proof is straightforward, and we leave it to the reader.

3) Obvious. ∎

Now let $(F, D, \{a, b_1, \dots, b_n\})$ be an admissible triple. Put $E = F((t_0))\dots((t_n))$. Suppose C is the division algebra over E such that

$$C \sim D \otimes (a, t_0) \otimes (b_1, t_1) \otimes \dots \otimes (b_n, t_n).$$

The main purpose of the present article is the following

PROPOSITION 9: 1) $\operatorname{ind} C = 2^{n+1}$.

2) *The algebra $C_{odd} \overset{def}{=} C_{E_{odd}}$ does not decompose into a tensor product of two nontrivial algebras over any maximal odd degree extension E_{odd} of E.*

Proof. 1) Since $C_L = 0$ we have $\operatorname{ind} C \leq 2^{n+1}$. On the other hand, applying Lemma 7 to the field $F_0 = F((t_0))$ and the algebra $C_0 = D \otimes (a, t_0)$ we get

$$\operatorname{ind} C = \operatorname{ind}[(D + (a, t_0)) + (b_1, t_1) + \dots + (b_n, t_n)] \geq 2^{n+1}.$$

2) The proof is by induction on n. Assume that the algebra C_{odd} is decomposable, $C_{odd} \simeq D_1 \otimes_{E_{odd}} D_2$ and $\operatorname{ind} D_1, \operatorname{ind} D_2 \geq 2$. Consider two cases:

a) $\operatorname{ind} D_1 = 2$ or $\operatorname{ind} D_2 = 2$.

Suppose for instance $\operatorname{ind} D_1 = 2$. By Lemma 5 $D_1 \simeq (pt_I, st_J)$, where

$$I, J \subset \{0, 1, \ldots, n\}, \quad p, s \in F^*.$$

Case a) splits in turn into two subcases.

a1) $I \neq \emptyset$ or $J \neq \emptyset$. Assume $I \neq \emptyset$.

Suppose $I = \{k+1, k+2, \ldots, n\}$ for some k (the general case can be treated similarly, using Lemma 7). Then we have

$$\begin{aligned} C_{E_{odd}(\sqrt{pt_I})} =& D + (a, t_0) + (b_1, t_1) + \cdots + (b_{n-1}, t_{n-1}) \\ &+ (b_n, pt_{k+1} \cdots t_{n-1}) \\ =& D + (b_n, p) + (a, t_0) + (b_1, t_1) + \cdots + (b_k, t_k) \\ &+ (b_{k+1} b_n, t_{k+1}) + \cdots + (b_{n-1} b_n, t_{n-1}). \end{aligned}$$

The field $E_{odd}(\sqrt{pt_I})$ can be considered as an iterated Laurent series field, namely, $E_{odd}(\sqrt{pt_I}) = F((t_0)) \ldots ((t_{n-1}))((x))$, where $x = \sqrt{pt_I}$. Put

$$L_1 = F(\sqrt{a}, \sqrt{b_1}, \ldots, \sqrt{b_k}, \sqrt{b_{k+1} b_n}, \ldots, \sqrt{b_{n-1} b_n}), \quad L_2 = F(\sqrt{b_n}).$$

Since $\operatorname{ind} C_{E_{odd}(\sqrt{pt_I})} = 1/2 \operatorname{ind} C = 2^n$, it follows by Lemma 7 that

$$(D + (b_n, p))_{L_1} = 0.$$

Therefore,

$$D = (D + (b_n, p)) + (b_n, p) \in {}_2\operatorname{Br}(L_1/F) + {}_2\operatorname{Br}(L_2/F),$$

which contradicts Corollary 4.

a2) $I = J = \emptyset$, i.e. $D_1 = (p, s)$, where $p, s \in F^*$.

Then

$$\operatorname{ind}[D + (p, s) + (a, t_0) + (b_1, t_1) + \cdots + (b_n, t_n)] = \operatorname{ind} D_2 = 2^n,$$

which contradicts Lemma 6.

Thus we have completed the proof in the case a). In particular, we have proved indecomposability of C in the case $n = 2$.

b) $\operatorname{ind} D_1 \geq 4$ and $\operatorname{ind} D_2 \geq 4$.

In particular, $n \geq 3$. Obviously, either $\operatorname{ind} {D_1}_{E_{odd}(\sqrt{a})} = \operatorname{ind} D_1$ or $\operatorname{ind} {D_2}_{E_{odd}(\sqrt{a})} = \operatorname{ind} D_2$. Assume that $\operatorname{ind} {D_1}_{E_{odd}(\sqrt{a})} = \operatorname{ind} D_1$. Since any finite extension of E_{odd} is a tower of quadratic extensions, there exists $c \in {E_{odd}}^*$ such that $\operatorname{ind} {D_1}_{E_{odd}(\sqrt{c})} = \frac{1}{2} \operatorname{ind} D_1$. Moreover, since F has no proper odd degree extensions, we may assume by Lemma 5 that $c = pt_I$ for some $p \in F^*$ and $I \subset \{0, 1, \ldots, n\}$. Let ${D_1}_{E_{odd}(\sqrt{pt_I})} \sim D_1'$, where D_1' is some nontrivial division algebra over $E_{odd}(\sqrt{pt_I})$. If $I \neq \emptyset$, we come to a contradiction just as in case a). So suppose that $I = \emptyset$. Since $\operatorname{ind} D_{1E_{odd}(\sqrt{a})} = \operatorname{ind} D_1$, we have $pa \notin F^{*2}$, and

$$\operatorname{ind} C_{E_{odd}(\sqrt{p})} = \operatorname{ind}[D + (a, t_0) + (b_1, t_1) + \cdots + (b_n, t_n)]_{E_{odd}(\sqrt{p})} = 2^n.$$

It follows from Lemma 6 that the elements $\overline{a}, \overline{b_1}, \ldots, \overline{b_n}$ are linearly dependent in $F(\sqrt{p})^*/F(\sqrt{p})^{*2}$. This in turn means that we may assume that either $p = b_I$ or $p = ab_I$ for some $\emptyset \neq I \subset \{1, \ldots, n\}$. Let $p = b_I$, for instance, where $I = \{k+1, \ldots, n\}$ (the general case can be treated with a slight modification, and is left to the reader). Put

$$K = F(\sqrt{p})((t_0)) \ldots ((t_{n-1})), \quad L = F(\sqrt{p})((t_0)) \ldots ((t_{n-1}))((\sqrt{t_n})).$$

Then

$$D \otimes (a, t_0) \otimes (b_1, t_1) \otimes \cdots \otimes (b_{n-1}, t_{n-1})_{L_{odd}} \sim C_{L_{odd}} \sim D'_1 \otimes_{L_{odd}} D_2$$

for any maximal odd degree extension L_{odd} of L, containing $E_{odd}(\sqrt{p})$. On the other hand, by Proposition 8 the triple $(F(\sqrt{p}), D_{F(\sqrt{p})}, \{a, b_1, \cdots, b_{n-1}\})$ is admissible. Hence by the induction hypothesis for any maximal odd degree extension K_{odd} of K we have

$$D \otimes (a, t_0) \otimes (b_1, t_1) \otimes \cdots \otimes (b_{n-1}, t_{n-1})_{K_{odd}} \sim C_{1K_{odd}},$$

C_1 being a division algebra over K such that it is indecomposable over K_{odd} and $\operatorname{ind} C_1 = 2^n$. By the first part of Lemma 5 we can choose K_{odd} in such a way that $K_{odd}((\sqrt{t_n})) \hookrightarrow L_{odd}$ over $K((\sqrt{t_n}))$. This implies

$$C_{1L_{odd}} \sim D'_1 \otimes_{L_{odd}} D_2.$$

Since, obviously, $\operatorname{ind}(D'_1 \otimes_{L_{odd}} D_2) = \operatorname{ind} C_{1L_{odd}} = 2^n$, we have

$$C_{1L_{odd}} \simeq D'_1 \otimes_{L_{odd}} D_2.$$

By ([Se], Chapter 2, Theorem 2) the field L_{odd} is the direct limit of fields $K_{odd}((u))$, where u is an indeterminate. Hence for some $u \in L_{odd}$ the algebras D'_1 and D_2 are defined over $K_{odd}((u))$ and

$$C_{1K_{odd}((u))} \simeq D'_1 \otimes_{K_{odd}((u))} D_2.$$

It is well known that for any field k and positive integer m not divided by $\operatorname{char} k$

$$H^2(k((x)), \mu_m) \simeq H^2(k, \mu_m) \oplus H^1(k, \mathbb{Z}/m\mathbb{Z}),$$

where μ_m is the group of mth roots of unity. From this and the cohomological interpretation of the Brauer group it follows that

$$D_1' = \operatorname{res}_{K_{odd}\left(\left(u^{\frac{1}{s}}\right)\right)/K_{odd}} \widetilde{D}_1, \quad D_2 = \operatorname{res}_{K_{odd}\left(\left(u^{\frac{1}{s}}\right)\right)/K_{odd}} \widetilde{D}_2$$

for a sufficiently large 2-power s and some central division algebras $\widetilde{D}_1$ and $\widetilde{D}_2$ over K_{odd}. Since the natural map $\operatorname{Br} K_{odd} \to \operatorname{Br} K_{odd}\left(\left(u^{\frac{1}{s}}\right)\right)$ is injective,

$$C_{1K_{odd}} = \widetilde{D_1} + \widetilde{D_2},$$

and by dimension count we conclude that

$$C_{1K_{odd}} \simeq \widetilde{D_1} \otimes_{K_{odd}} \widetilde{D_2},$$

a contradiction to indecomposability of the algebra C_1 over K_{odd}. ∎

Summing up the obtained results and changing the notation a bit we get the following

COROLLARY 10: *Let k be a field, $n \geq 3$. Suppose the elements $a_1, \ldots, a_n \in k^*/k^{*2}$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$. Then there exists a field extension F/k, a quaternion algebra $D \in {}_2\operatorname{Br} F$ and a division algebra $C \in {}_2\operatorname{Br} F((t_1))\ldots((t_n))$ of index 2^n such that*

1) *C is indecomposable over any odd degree extension of $E = F((t_1))\ldots((t_n))$.*
2) *$M_2(C) \simeq (a_1, t_1) \otimes_E \cdots \otimes_E (a_n, t_n) \otimes_E D$.*
3) *$D \in {}_2\operatorname{Br}(F(\sqrt{a_1}, \ldots, \sqrt{a_n})/F)$.*

Proof. This immediately follows from Propositions 2, 8 and 9. ∎

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